

Predicted value of $0\nu\beta\beta$ -decay effective Majorana mass with error of lightest neutrino mass

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Abstract

Assuming that the lightest neutrino mass m_0 is measured, we study the influence of error of the measured m_0 on the uncertainty of the predicted value of the neutrinoless double beta decay ($0\nu\beta\beta$) effective Majorana mass $|m_{ee}|$. The error of the predicted value of effective Majorana mass $\sigma(|m_{ee}|)$ due to the error $\sigma(m_0)$ is obtained by use of the propagation of errors. We investigate in detail how $\sigma(|m_{ee}|)$ behaves when the Majorana phases β and α change, and also when the size of the lightest neutrino mass changes. It is shown that $\sigma(|m_{ee}|)$ does not exceed $\sigma(m_0)$, and that the minimum value of $\sigma(|m_{ee}|)$ in the $\beta - \alpha$ plane can be zero if the size of m_0 is rather small. In the normal mass ordering case, the maximum value of $\sigma(|m_{ee}|)$ in the $\beta - \alpha$ plane becomes about $0.68\sigma(m_0)$ if $m_0(=m_1)$ is very small, while in the inverted mass ordering case, it becomes about $0.02\sigma(m_0)$ if $m_0(=m_3)$ is very small. By use of the calculated $\sigma(|m_{ee}|)$, we also discuss the minimum condition for the relative error of the lightest neutrino mass to raise the possibility of obtaining the information on β and α .

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1 Introduction

The experiments of neutrinoless double beta decay ($0\nu\beta\beta$) play an important role in judging whether the neutrinos are Majorana particles or Dirac particles. The neutrinoless double beta decay is the double beta decay which does not emit any neutrinos. If we assume $0\nu\beta\beta$ is caused by the exchange of three light Majorana neutrinos, the observation of $0\nu\beta\beta$ is the evidence that the neutrinos are Majorana fermions [1, 2, 3]. Many experiments of $0\nu\beta\beta$ are in progress and planned: CANDLES [4], NEMO-3 [5], SOLOTVINO [6], CUORICINO [7], EXO-200 [8], KamLAND-Zen [9], etc. The signal of $0\nu\beta\beta$ has not been detected yet. If neutrinos are Majorana fermions, the lepton mixing matrix U (MNS matrix [10]) is expressed by six parameters; three lepton mixing angles ($\theta_{12}, \theta_{23}, \theta_{13}$), CP violating Dirac phase δ , and two CP violating Majorana phases α, β . The two Majorana phases α, β are the degrees of freedom arising from the fact that the particle and the antiparticle are the same in Majorana fermion. Among these six parameters, the value of three lepton mixing angles ($\theta_{12}, \theta_{23}, \theta_{13}$) are measured [12]. Especially, the measurement of θ_{13} was made recently [13], whose effect on the study of $0\nu\beta\beta$ had been investigated [14]. The values of Dirac phase δ and two Majorana phases α, β are unknown to this day. If we suppose that $0\nu\beta\beta$ is caused by the exchange of three light Majorana neutrinos, the amplitude of the decay is proportional to the $0\nu\beta\beta$ -decay effective Majorana mass $|m_{ee}|$,

$$|m_{ee}| = \left| \sum_{i=1}^3 m_i U_{ei}^2 \right| = \left| m_1 |U_{e1}|^2 + m_2 |U_{e2}|^2 e^{2i\alpha} + m_3 |U_{e3}|^2 e^{2i\beta} \right|, \quad (1)$$

where $m_i (i = 1, 2, 3)$ is neutrino mass of i -th mass eigenstate. The unitary matrix U is the lepton mixing matrix (MNS matrix) and parametrized as follows,

$$U = \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_{23} & s_{23} \\ 0 & -s_{23} & c_{23} \end{pmatrix} \begin{pmatrix} c_{13} & 0 & s_{13}e^{-i\delta} \\ 0 & 1 & 0 \\ -s_{13}e^{i\delta} & 0 & c_{13} \end{pmatrix} \begin{pmatrix} c_{12} & s_{12} & 0 \\ -s_{12} & c_{12} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{i\alpha} & 0 \\ 0 & 0 & e^{i\beta} \end{pmatrix}, \quad (2)$$

where s_{ij} and c_{ij} are sine and cosine of the lepton mixing angle θ_{ij} , respectively. The $U_{ei} (i = 1, 2, 3)$ satisfy $|U_{e1}|^2 + |U_{e2}|^2 + |U_{e3}|^2 = 1$, and we shall restrict the Majorana phases, $-\pi/2 < \beta, \alpha \leq \pi/2$, which are enough to investigate the variation of $|m_{ee}|$.

The effective mass $|m_{ee}|$ depends on seven parameters; two mixing angles θ_{12} and θ_{13} , three neutrino masses m_1, m_2, m_3 , and two Majorana phases β, α . With regard to these seven parameters, the following four quantities are measured by experiments; i.e., two mixing angles θ_{12}, θ_{13} , and two mass squared differences $\Delta m_{\odot}^2 \equiv m_2^2 - m_1^2$, $\Delta m_{\text{A}}^2 \equiv |m_3^2 - m_1^2| \approx |m_3^2 - m_2^2|$. Two Majorana phases β, α (if Majorana particles) and the absolute neutrino mass scale (we can take the lightest neutrino mass m_0 as the absolute neutrino mass scale) are not measured, though an upper limit for the sum of the three light neutrino mass has been reported [15, 16]. There remains two possibilities with respect to the order of m_1, m_2 , and m_3 , that is the normal mass ordering ($m_3 > m_2 > m_1$) and inverted mass ordering ($m_2 > m_1 > m_3$). In the normal

mass ordering, we take the lightest neutrino mass $m_0 = m_1$ as the absolute neutrino mass scale,

$$\begin{aligned} m_2 &= \sqrt{m_1^2 + \Delta m_{\odot}^2}, \\ m_3 &= \sqrt{m_1^2 + \Delta m_{\text{A}}^2}. \end{aligned} \quad (3)$$

In the inverted mass ordering, we take the lightest neutrino mass $m_0 = m_3$ as the absolute neutrino mass scale,

$$\begin{aligned} m_2 &= \sqrt{m_3^2 + \Delta m_{\odot}^2 + \Delta m_{\text{A}}^2}, \\ m_1 &= \sqrt{m_3^2 + \Delta m_{\text{A}}^2}. \end{aligned} \quad (4)$$

The signal of $0\nu\beta\beta$ has not been detected, but an upper limit of the effective mass $|m_{ee}|$ is obtained. While information on Majorana phases β, α is obtained by $0\nu\beta\beta$ experiments [17, 18, 19, 20], it is notable that the other experiments measuring the absolute neutrino mass scale are also important to determine the phases β, α as discussed in Ref [19, 20]. One of the most often used method for studying $0\nu\beta\beta$ is that, regarding the lightest neutrino mass m_0 as a free parameter, the predicted values of the effective mass $|m_{ee}|$ are investigated for each given value of the lightest neutrino mass. For instance, the authors in Ref [21] calculated 1σ error on the predicted value of $|m_{ee}|$ due to the uncertainties in the values of the neutrino oscillation parameters θ_{12}, θ_{13} , Δm_{\odot}^2 , and Δm_{A}^2 .

We here suppose that the lightest neutrino mass m_0 is measured,

$$m_0 \pm \sigma(m_0), \quad (5)$$

where $\sigma(m_0)$ is 1σ error on m_0 . If the normal mass ordering is assumed, we have $m_0 = m_1$; while if the inverted mass ordering, $m_0 = m_3$. In this paper, we study contributions of the error on the lightest neutrino mass m_0 to the uncertainty of the predicted value of the Majorana effective mass $|m_{ee}|$ in the normal mass ordering case and in the inverted mass ordering case, respectively. In order to predict the value of $|m_{ee}|$ which depends on the Majorana phases β and α , we need the values of the absolute neutrino mass scale (we can take the lightest neutrino mass m_0) and four oscillation parameters; θ_{12}, θ_{13} , Δm_{\odot}^2 , and Δm_{A}^2 . These four oscillation parameters have been measured with relatively small errors, respectively. Since we focus on the uncertainty $\sigma(m_0)$ of the lightest neutrino mass, we do not take into account the uncertainties of θ_{12}, θ_{13} , Δm_{\odot}^2 , and Δm_{A}^2 in this article. By use of the propagation of errors, the 1σ error on $|m_{ee}|$ due to the uncertainty of m_0 is obtained,

$$\sigma(|m_{ee}|)^2 = \left\{ \frac{\partial |m_{ee}|}{\partial m_0} \sigma(m_0) \right\}^2. \quad (6)$$

The predicted value of $|m_{ee}|$ depends on the Majorana phases β, α , and if the uncertainty $\sigma(|m_{ee}|)$ is large, the dependence of $|m_{ee}|$ on β and α is obscured. To obtain

some information on the Majorana phases β and α through the effective mass, the uncertainty $\sigma(|m_{ee}|)$ should not be large. We study how the uncertainty $\sigma(|m_{ee}|)$ due to the uncertainty $\sigma(m_0)$ behaves in the $\beta - \alpha$ plane, and how it changes according to the size of the lightest neutrino mass m_0 . We also discuss the necessary condition for the relative error $\sigma(m_0)/m_0$ of the lightest neutrino mass to raise the possibility of probing the Majorana phases.

The paper is organized as follows. In section 2, the normal mass ordering is assumed, so that the lightest neutrino mass $m_0 = m_1$. The 1σ error $\sigma(|m_{ee}|)$ on the predicted value of the effective mass $|m_{ee}|$ is obtained by use of the propagation of errors, Eq.(6). The result of straightforward calculation of $\sigma(|m_{ee}|)$ is so complicated that we introduce complex-valued quantities P and Q in subsection 2.1. Expressing $\sigma(|m_{ee}|)$ by these complex-valued quantities P and Q , one can analyze it easily. We then investigate how $\sigma(|m_{ee}|)$ behaves in the $\beta - \alpha$ plane for each size of the lightest neutrino mass m_1 . In section 3, the inverted mass ordering is assumed, so that the lightest neutrino mass $m_0 = m_3$. The 1σ error $\sigma(|m_{ee}|)$ in the inverted mass ordering case is studied in the same manner as section 2. In section 4, we discuss the necessary condition that the relative error of the lightest neutrino mass should satisfy in order to raise the possibility of probing the Majorana phases β, α . Section 5 is devoted to conclusions.

2 Normal mass ordering case ($m_3 > m_2 > m_1$)

In this chapter, the normal mass ordering ($m_3 > m_2 > m_1$) is assumed and the lightest neutrino mass $m_0 = m_1$. We assume that the lightest neutrino mass m_1 is measured,

$$m_1 \pm \sigma(m_1), \quad (7)$$

where $\sigma(m_1)$ is the 1σ error on m_1 . The 1σ error $\sigma(|m_{ee}|)$ of the predicted value of $|m_{ee}|$ due to the uncertainty $\sigma(m_1)$ will be calculated and studied below. As discussed in section 1, we do not consider the uncertainties of four oscillation parameters, $\theta_{12}, \theta_{13}, \Delta m_{\odot}^2$, and Δm_{A}^2 and therefore the $\sigma(|m_{ee}|)$ is obtained by Eq.(6),

$$\sigma(|m_{ee}|) = \left| \frac{\partial |m_{ee}|}{\partial m_1} \right| \sigma(m_1). \quad (8)$$

2.1 Calculation of $\sigma(|m_{ee}|)$

The explicit calculation of the factor $\partial |m_{ee}|/\partial m_1$ in the right-handed side of Eq.(8) shows

$$\begin{aligned} \frac{\partial |m_{ee}|}{\partial m_1} = \frac{m_1}{|m_{ee}|} & \left[|U_{e1}|^4 + |U_{e2}|^4 + |U_{e3}|^4 + \left(\frac{m_2}{m_1} + \frac{m_1}{m_2} \right) |U_{e1}|^2 |U_{e2}|^2 \cos 2\alpha \right. \\ & + \left(\frac{m_3}{m_1} + \frac{m_1}{m_3} \right) |U_{e3}|^2 |U_{e1}|^2 \cos 2\beta \\ & \left. + \left(\frac{m_3}{m_2} + \frac{m_2}{m_3} \right) |U_{e2}|^2 |U_{e3}|^2 (\cos 2\alpha \cos 2\beta + \sin 2\alpha \sin 2\beta) \right], \quad (9) \end{aligned}$$

where

$$\begin{aligned} |m_{ee}| = & \left[(m_1 |U_{e1}|^2)^2 + (m_2 |U_{e2}|^2)^2 + (m_3 |U_{e3}|^2)^2 \right. \\ & + 2(m_1 |U_{e1}|^2)(m_2 |U_{e2}|^2) \cos 2\alpha \\ & + 2(m_2 |U_{e2}|^2)(m_3 |U_{e3}|^2) (\cos 2\alpha \cos 2\beta + \sin 2\alpha \sin 2\beta) \\ & \left. + 2(m_3 |U_{e3}|^2)(m_1 |U_{e1}|^2) \cos 2\beta \right]^{\frac{1}{2}}. \quad (10) \end{aligned}$$

We prefer to study the behavior of $\sigma(|m_{ee}|)$ in the $\beta - \alpha$ plane analytically. When it is difficult to study it analytically, the numerical calculations will be carried out in which the following reference values of four oscillation parameters are used [12],

$$\begin{aligned} \Delta m_\odot^2 &= 7.50 \times 10^{-5} \text{ eV}^2, \\ \Delta m_A^2 &= 2.473 \times 10^{-3} \text{ eV}^2, \\ \sin^2 \theta_{12} &= 0.302, \\ \sin^2 \theta_{13} &= 0.0227. \end{aligned} \quad (11)$$

These reference values lead to

$$\begin{aligned} |U_{e1}|^2 &= c_{12}^2 c_{13}^2 = 0.682, \\ |U_{e2}|^2 &= s_{12}^2 c_{13}^2 = 0.295, \\ |U_{e3}|^2 &= s_{13}^2 = 0.0227, \end{aligned} \quad (12)$$

and one has $|U_{e1}|^2 > |U_{e2}|^2 \gg |U_{e3}|^2$ and $\Delta m_A^2 \gg \Delta m_\odot^2$. For an arbitrary value of m_1 , the relation $m_3 |U_{e3}|^2 < m_2 |U_{e2}|^2$ is supposed in this section.

Although the analysis of the calculated result Eq.(9) of the factor $\partial|m_{ee}|/\partial m_1$ is very hard, we can analyze it more easily by introducing the following two complex quantities P and Q ,

$$P \equiv m_1 |U_{e1}|^2 + m_2 |U_{e2}|^2 e^{2i\alpha} + m_3 |U_{e3}|^2 e^{2i\beta}, \quad (13)$$

$$Q \equiv \frac{1}{m_1} |U_{e1}|^2 + \frac{1}{m_2} |U_{e2}|^2 e^{2i\alpha} + \frac{1}{m_3} |U_{e3}|^2 e^{2i\beta}, \quad (14)$$

where m_2 and m_3 are given by Eq.(3). Using the relation $\partial P / \partial m_1 = m_1 Q$, one has

$$\frac{\partial |m_{ee}|}{\partial m_1} = \frac{m_1}{|P|} \text{Re}[PQ^*] = m_1 |Q^*| \frac{\text{Re}[PQ^*]}{|PQ^*|} = m_1 |Q| \cos(\text{Arg}(PQ^*)), \quad (15)$$

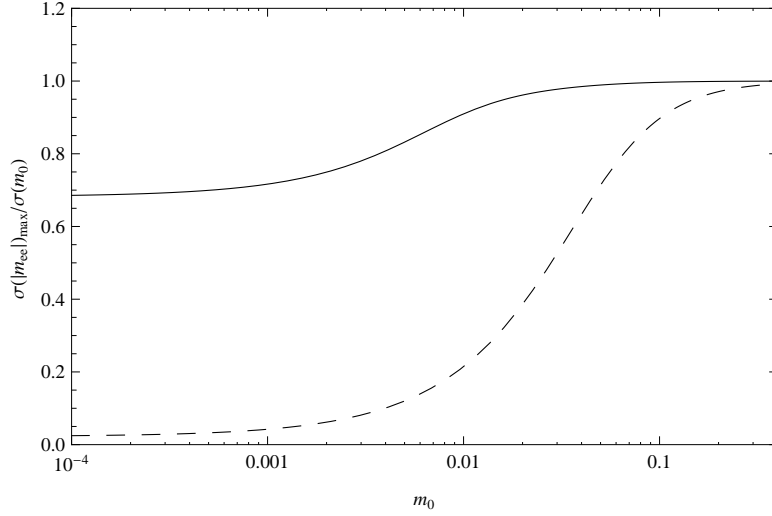


Figure 1: The value of $\sigma(|m_{ee}|)_{\max}/\sigma(m_0)$ as a function of the lightest neutrino mass m_0 (in eV). The solid line represents the normal mass ordering case ($m_0 = m_1$), and the long-dashed line represents the inverted mass ordering case ($m_0 = m_3$), respectively.

in which $m_1|Q| \neq 0$ because of $m_3 > m_2 > m_1$. The absolute value of the factor $\partial|m_{ee}|/\partial m_1$ satisfies

$$\left| \frac{\partial|m_{ee}|}{\partial m_1} \right| \leq |U_{e1}|^2 + \frac{m_1}{m_2}|U_{e2}|^2 + \frac{m_1}{m_3}|U_{e3}|^2 < 1, \quad (16)$$

since $|\cos(\text{Arg}(PQ^*))| \leq 1$. Now, we denote the maximum value of $\sigma(|m_{ee}|)$ in the $\beta - \alpha$ plane as $\sigma(|m_{ee}|)_{\max}$, and the minimum value as $\sigma(|m_{ee}|)_{\min}$. From Eq.(16), the maximum value $\sigma(|m_{ee}|)_{\max}$ in the $\beta - \alpha$ plane is less than the value of $\sigma(m_1)$,

$$\sigma(|m_{ee}|)_{\max} = \left\{ |U_{e1}|^2 + \frac{m_1}{\sqrt{m_1^2 + \Delta m_{\odot}^2}} |U_{e2}|^2 + \frac{m_1}{\sqrt{m_1^2 + \Delta m_{\text{A}}^2}} |U_{e3}|^2 \right\} \sigma(m_1). \quad (17)$$

The coefficient of $\sigma(m_1)$ in the right-handed side of Eq.(17) is a monotone increasing function of the lightest neutrino mass m_1 . The dependence of $\sigma(|m_{ee}|)_{\max}/\sigma(m_1)$ on m_1 is shown in Fig.1 in which the lightest neutrino mass $m_0 = m_1$ in the normal mass ordering. As seen from the figure, the ratio $\sigma(|m_{ee}|)_{\max}/\sigma(m_1)$ is almost equal to 1 when the lightest neutrino mass m_1 is large, whereas it is equal to the value $|U_{e1}|^2 \approx 0.68$ when m_1 is approximately zero. The case of the inverted mass ordering in Fig.1 will be discussed in section 3. The minimum value $\sigma(|m_{ee}|)_{\min}$ in the $\beta - \alpha$ plane can take the value zero if $\text{Re}[PQ^*]/|PQ^*| = \cos(\text{Arg}(PQ^*)) = 0$ is realized.

2.2 Behavior of $\sigma(|m_{ee}|)$ according to the size of m_1 .

In this subsection, the behavior of $\sigma(|m_{ee}|)$ in the $\beta - \alpha$ plane is investigated according to the size of the lightest neutrino mass m_1 in the normal mass ordering case. The expression of the maximum value $\sigma(|m_{ee}|)_{\max}$ is obtained in the preceding subsection. We are especially interested in the case where $\sigma(|m_{ee}|)$ takes the value zero or approximately zero in the $\beta - \alpha$ plane, since the uncertainty of the predicted value of $|m_{ee}|$ due to the uncertainty of the lightest neutrino mass becomes zero in that case. The behavior of $\sigma(|m_{ee}|)$ will be studied according to the size of m_1 in the following (A)(D).

2.2.1 (A): $m_1 \rightarrow 0$

In this case, the upper limit of the measured value of the lightest neutrino mass is almost zero. Although $\sigma(m_1) \rightarrow 0$ would be impossible in practice, we shall study this case because of the characteristic behavior of $\partial|m_{ee}|/\partial m_1$ in the limit $m_1 \rightarrow 0$,

$$\frac{\partial|m_{ee}|}{\partial m_1} \rightarrow |U_{e1}|^2 \frac{\text{Re}(P)}{\sqrt{\{\text{Re}(P)\}^2 + \{\text{Im}(P)\}^2}}. \quad (18)$$

On the plane curve in the $\beta - \alpha$ plane,

$$\text{Re}(P) = \sqrt{\Delta m_{\odot}^2} |U_{e2}|^2 \cos 2\alpha + \sqrt{\Delta m_{\text{A}}^2} |U_{e3}|^2 \cos 2\beta = 0, \quad (19)$$

or

$$\cos 2\alpha + 0.442 \cos 2\beta = 0, \quad (20)$$

the factor $\partial|m_{ee}|/\partial m_1$ in Eq.(18) takes the minimum value zero,

$$\left(\frac{\partial|m_{ee}|}{\partial m_1} \right)_{\min} = \left(\frac{\partial|m_{ee}|}{\partial m_1} \right)_{\text{Re}(P)=0} = 0. \quad (21)$$

On the other side, on the plane curve in the $\beta - \alpha$ plane,

$$\text{Im}(P) = \sqrt{\Delta m_{\odot}^2} |U_{e2}|^2 \sin 2\alpha + \sqrt{\Delta m_{\text{A}}^2} |U_{e3}|^2 \sin 2\beta = 0, \quad (22)$$

or

$$\sin 2\alpha + 0.442 \sin 2\beta = 0, \quad (23)$$

the factor takes the maximum value,

$$\left(\frac{\partial|m_{ee}|}{\partial m_1} \right)_{\max} = \left(\frac{\partial|m_{ee}|}{\partial m_1} \right)_{\text{Im}(P)=0} = |U_{e1}|^2 = 0.682. \quad (24)$$

2.2.2 (B): $|U_{e1}|^2/m_1 \gg |U_{e2}|^2/m_2$ ($m_1 \ll \sqrt{\Delta m_\odot^2} |U_{e1}|^2/|U_{e2}|^2 \sim 0.020 \text{ eV}$)

For $m_1 \ll 0.020 \text{ eV}$, Q can be approximated by

$$Q \approx \frac{1}{m_1} |U_{e1}|^2, \quad (25)$$

which leads to

$$\frac{\partial |m_{ee}|}{\partial m_1} \approx (m_1 Q) \frac{\text{Re}(P)}{|P|} \approx |U_{e1}|^2 \frac{\text{Re}(P)}{\sqrt{\{\text{Re}(P)\}^2 + \{\text{Im}(P)\}^2}}. \quad (26)$$

On the plane curve in the $\beta - \alpha$ plane,

$$\text{Re}(P) = m_1 |U_{e1}|^2 + m_2 |U_{e2}|^2 \cos 2\alpha + m_3 |U_{e3}|^2 \cos 2\beta = 0, \quad (27)$$

or

$$0.682 m_1 + 0.295 \sqrt{m_1^2 + \Delta m_\odot^2} \cos 2\alpha + 0.0227 \sqrt{m_1^2 + \Delta m_\text{A}^2} \cos 2\beta = 0, \quad (28)$$

the factor becomes approximately zero $\partial |m_{ee}|/\partial m_1 \approx 0$, and we have

$$\sigma(|m_{ee}|)_{\min} = \sigma(|m_{ee}|)_{\text{Re}(P)=0} \approx 0. \quad (29)$$

From the equation of the above plane curve Eq.(27), $\cos 2\alpha$ can be expressed as $\cos 2\alpha = (-m_1 |U_{e1}|^2 - m_3 |U_{e3}|^2 \cos 2\beta)/(m_2 |U_{e2}|^2)$, which leads to

$$\frac{-m_1 |U_{e1}|^2 - m_3 |U_{e3}|^2}{m_2 |U_{e2}|^2} \leq \cos 2\alpha \leq \frac{-m_1 |U_{e1}|^2 + m_3 |U_{e3}|^2}{m_2 |U_{e2}|^2}. \quad (30)$$

If m_1 satisfies $(-m_1 |U_{e1}|^2 + m_3 |U_{e3}|^2)/(m_2 |U_{e2}|^2) < -1$, that is, $0.0063 \text{ eV} < m_1$, two Majorana phases (β, α) satisfying $\text{Re}(P) = 0$ do not exist. Actually, the numerical calculation shows that, for $0.0063 \text{ eV} \lesssim m_1$, there exists no plane curve in the $\beta - \alpha$ plane which satisfies $\sigma(|m_{ee}|) = 0$.

On the other hand, on the plane curve in the $\beta - \alpha$ plane,

$$\text{Im}(P) = m_2 |U_{e2}|^2 \sin 2\alpha + m_3 |U_{e3}|^2 \sin 2\beta = 0, \quad (31)$$

or

$$0.295 \sqrt{m_1^2 + \Delta m_\odot^2} \sin 2\alpha + 0.0227 \sqrt{m_1^2 + \Delta m_\text{A}^2} \sin 2\beta = 0, \quad (32)$$

the factor $\partial |m_{ee}|/\partial m_1$ takes the maximum value $|U_{e1}|^2$ and thereby,

$$\sigma(|m_{ee}|)_{\max} = \sigma(|m_{ee}|)_{\text{Im}(P)=0} \approx |U_{e1}|^2 \sigma(m_1) = 0.682 \sigma(m_1). \quad (33)$$

2.2.3 (C): $m_2|U_{e2}|^2 \gg m_3|U_{e3}|^2$ **and** $m_1^2 \lesssim \Delta m_A^2$ ($0.01 \text{ eV} \lesssim m_1 \lesssim 0.05 \text{ eV}$)

In this case, we obtain the following relation, $|U_{e2}|^2 \gg (m_3/m_2)|U_{e3}|^2 > (m_2/m_3)|U_{e3}|^2$, which allows us to use the approximation,

$$\begin{aligned} P &\approx m_1|U_{e1}|^2 + m_2|U_{e2}|^2 e^{2i\alpha}, \\ Q &\approx \frac{1}{m_1}|U_{e1}|^2 + \frac{1}{m_2}|U_{e2}|^2 e^{2i\alpha}. \end{aligned} \quad (34)$$

The factor $\partial|m_{ee}|/\partial m_1$ is then approximated,

$$\frac{\partial|m_{ee}|}{\partial m_1} \approx \frac{m_1 \left\{ |U_{e1}|^4 + |U_{e2}|^4 + \left(\frac{m_2}{m_1} + \frac{m_1}{m_2} \right) |U_{e1}|^2 |U_{e2}|^2 \cos 2\alpha \right\}}{(m_1|U_{e1}|^2)^2 + (m_2|U_{e2}|^2)^2 + 2(m_1|U_{e1}|^2)(m_2|U_{e2}|^2) \cos 2\alpha}. \quad (35)$$

The factor $\partial|m_{ee}|/\partial m_1$ scarcely depends on the Majorana phase β , and is a monotone decreasing function of the Majorana phase α ($0 \leq \alpha \leq \pi/2$). The $\partial|m_{ee}|/\partial m_1$ takes the maximum value $|U_{e1}|^2 + (m_1/m_2)|U_{e2}|^2 + O(|U_{e3}|^2)$ at $\alpha = 0$, and the minimum value $|U_{e1}|^2 - (m_1/m_2)|U_{e2}|^2 + O(|U_{e3}|^2)$ at $\alpha = \pi/2$. Namely, $\sigma(|m_{ee}|)$ scarcely depends on β , and is a monotone decreasing function of α ($0 \leq \alpha \leq \pi/2$). The $\sigma(|m_{ee}|)$ takes the maximum value at $\alpha = 0$,

$$\begin{aligned} \sigma(|m_{ee}|)_{\max} &= \sigma(|m_{ee}|)_{\alpha=0} \\ &\approx \left\{ |U_{e1}|^2 + \frac{m_1}{m_2}|U_{e2}|^2 \right\} \sigma(m_1) \\ &= \left\{ 0.682 + 0.295 \frac{m_1}{\sqrt{m_1^2 + \Delta m_\odot^2}} \right\} \sigma(m_1), \end{aligned} \quad (36)$$

and the minimum value at $\alpha = \pi/2$,

$$\begin{aligned} \sigma(|m_{ee}|)_{\min} &= \sigma(|m_{ee}|)_{\alpha=\pi/2} \\ &\approx \left\{ |U_{e1}|^2 - \frac{m_1}{m_2}|U_{e2}|^2 \right\} \sigma(m_1) \\ &= \left\{ 0.682 - 0.295 \frac{m_1}{\sqrt{m_1^2 + \Delta m_\odot^2}} \right\} \sigma(m_1), \end{aligned} \quad (37)$$

where we have neglected the term of the order $O(|U_{e3}|^2)$ in $\partial|m_{ee}|/\partial m_1$.

2.2.4 (D): $m_1^2 \gg \Delta m_A^2$ **such that** $m_1 \approx m_2 \approx m_3$ ($m_1^2 \gg (0.05 \text{ eV})^2$)

In this case, using the relation $Q \approx P/m_1^2$, one has

$$\frac{\partial|m_{ee}|}{\partial m_1} \approx \frac{|m_{ee}|}{m_1} \approx ||U_{e1}|^2 + |U_{e2}|^2 e^{2i\alpha} + |U_{e3}|^2 e^{2i\beta}|, \quad (38)$$

or

$$\frac{\sigma(|m_{ee}|)}{|m_{ee}|} \approx \frac{\sigma(m_1)}{m_1}, \quad (39)$$

which means that the relative error of the effective mass $|m_{ee}|$ has the same magnitude as that of the lightest neutrino mass m_1 . If the relative error of m_1 is fixed, $\sigma(|m_{ee}|)$ is proportional to $|m_{ee}|$. The $|m_{ee}|$ takes the maximum value $m_1\{|U_{e1}|^2 + |U_{e2}|^2 + |U_{e3}|^2\} = m_1$ in the $\beta - \alpha$ plane at the point $(\beta, \alpha) = (0, 0)$, therefore $\sigma(|m_{ee}|)$ takes the maximum value,

$$\sigma(|m_{ee}|)_{\max} = \sigma(|m_{ee}|)_{(\beta, \alpha)=(0, 0)} \approx \sigma(m_1), \quad (40)$$

at the same point $(\beta, \alpha) = (0, 0)$. On the other side, $|m_{ee}|$ takes the minimum value $m_1\{|U_{e1}|^2 - |U_{e2}|^2 - |U_{e3}|^2\}$ in the $\beta - \alpha$ plane at the point $(\beta, \alpha) = (\pi/2, \pi/2)$, therefore $\sigma(|m_{ee}|)$ takes the minimum value,

$$\begin{aligned} \sigma(|m_{ee}|)_{\min} &= \sigma(|m_{ee}|)_{(\beta, \alpha)=(\pi/2, \pi/2)} \\ &\approx \left\{ |U_{e1}|^2 - |U_{e2}|^2 - |U_{e3}|^2 \right\} \sigma(m_1) \\ &= 0.364 \sigma(m_1), \end{aligned} \quad (41)$$

at the same point $(\beta, \alpha) = (\pi/2, \pi/2)$.

3 Inverted mass ordering case; ($m_2 > m_1 > m_3$)

In this chapter, the inverted mass ordering ($m_2 > m_1 > m_3$) is supposed and the lightest neutrino mass $m_0 = m_3$. We assume that the lightest neutrino mass m_3 is measured,

$$m_3 \pm \sigma(m_3), \quad (42)$$

where $\sigma(m_3)$ is the 1σ error on m_3 . The 1σ error $\sigma(|m_{ee}|)$ of the predicted value of $|m_{ee}|$ due to the uncertainty $\sigma(m_3)$ is obtained from Eq.(6),

$$\sigma(|m_{ee}|) = \left| \frac{\partial |m_{ee}|}{\partial m_3} \right| \sigma(m_3), \quad (43)$$

where we have not considered the uncertainties of four oscillation parameters, $\theta_{12}, \theta_{13}, \Delta m_{\odot}^2$, and Δm_A^2 as in section 2.

3.1 Calculation of $\sigma(|m_{ee}|)$

The explicit calculation of the factor $\partial |m_{ee}| / \partial m_3$ shows that the expression of it is very complicated as in the normal mass ordering case, and we shall not present here the calculated result. As in section 2, we prefer to study the behavior of $\sigma(|m_{ee}|)$ in the $\beta - \alpha$ plane analytically also in the inverted mass ordering case. When it is difficult to study it analytically, the numerical calculations will be carried out in which the

reference values Eq.(11) and Eq.(12) are also used in this chapter. In the inverted mass ordering, the relation $m_1|U_{e1}|^2 > m_2|U_{e2}|^2 \gg m_3|U_{e3}|^2$ holds for an arbitrary value of the lightest neutrino mass m_3 . Furthermore, the following relation is satisfied,

$$1 < \frac{m_2}{m_1} = \sqrt{1 + \frac{\Delta m_\odot^2}{m_3^2 + \Delta m_A^2}} < \sqrt{1 + \frac{\Delta m_\odot^2}{\Delta m_A^2}} \approx 1.015, \quad (44)$$

thereby, we use the approximation $m_1 \approx m_2$. While the expression of $\partial|m_{ee}|/\partial m_3$ calculated explicitly is very complicated, we can analyze $\partial|m_{ee}|/\partial m_3$ more easily by introducing the following complex quantities P and Q , as in section 2,

$$P \equiv m_1|U_{e1}|^2 + m_2|U_{e2}|^2 e^{2i\alpha} + m_3|U_{e3}|^2 e^{2i\beta}, \quad (45)$$

$$Q \equiv \frac{1}{m_1}|U_{e1}|^2 + \frac{1}{m_2}|U_{e2}|^2 e^{2i\alpha} + \frac{1}{m_3}|U_{e3}|^2 e^{2i\beta}, \quad (46)$$

where m_1 and m_2 are given by Eq.(4). It should be noted that (m_1, m_2, m_3) of the inverted mass ordering case is different from (m_1, m_2, m_3) of the normal mass ordering case. Using the relation, $\partial P/\partial m_3 = m_3 Q$, we have

$$\frac{\partial|m_{ee}|}{\partial m_3} = \frac{m_3}{|P|} \text{Re}[PQ^*] = m_3|Q| \cos(\text{Arg}(PQ^*)), \quad (47)$$

which leads to

$$\left| \frac{\partial|m_{ee}|}{\partial m_3} \right| \leq \frac{m_3}{m_1}|U_{e1}|^2 + \frac{m_3}{m_2}|U_{e2}|^2 + |U_{e3}|^2 < 1. \quad (48)$$

Here, we denote the maximum value of $\sigma(|m_{ee}|)$ in the $\beta - \alpha$ plane as $\sigma(|m_{ee}|)_{\text{max}}$, and the minimum value as $\sigma(|m_{ee}|)_{\text{min}}$. From Eq.(48), $\sigma(|m_{ee}|)_{\text{max}}$ is less than $\sigma(m_3)$,

$$\sigma(|m_{ee}|)_{\text{max}} = \left\{ \frac{m_3}{\sqrt{m_3^2 + \Delta m_A^2}}|U_{e1}|^2 + \frac{m_3}{\sqrt{m_3^2 + \Delta m_\odot^2 + \Delta m_A^2}}|U_{e2}|^2 + |U_{e3}|^2 \right\} \sigma(m_3). \quad (49)$$

The coefficient of $\sigma(m_3)$ in the right-handed side of Eq.(49) is a monotone increasing function of the lightest neutrino mass m_3 . The dependence of $\sigma(|m_{ee}|)_{\text{max}}/\sigma(m_3)$ on m_3 is shown in Fig.1 in which the lightest neutrino mass $m_0 = m_3$ in the inverted mass ordering. In the inverted mass ordering case, the ratio of the maximum value of $\sigma(|m_{ee}|)$ in the $\beta - \alpha$ plane to $\sigma(m_3)$, $\sigma(|m_{ee}|)_{\text{max}}/\sigma(m_3)$, is almost equal to 1 when the lightest neutrino mass m_3 is large such as $m_3^2 \gg \Delta m_A^2$. When the value of m_3 decreases, the value of the ratio $\sigma(|m_{ee}|)_{\text{max}}/\sigma(m_3)$ decreases rapidly and it becomes $|U_{e3}|^2 \approx 0.02$ if m_3 is approximately zero. On the other hand, in the normal mass ordering case discussed in section 2, the ratio $\sigma(|m_{ee}|)_{\text{max}}/\sigma(m_1)$ is almost 1 when the lightest neutrino mass m_1 is large such as $m_1^2 \gg \Delta m_A^2$. When the value of m_1 decreases, however, the value of $\sigma(|m_{ee}|)_{\text{max}}/\sigma(m_1)$ decreases gradually and it becomes $|U_{e1}|^2 \approx 0.68$ if m_1 is approximately zero. In the case of large value of the

lightest neutrino mass $m_0^2 \gg \Delta m_A^2$, the above results can be easily understood since the difference between the normal mass ordering and the inverted mass ordering disappears when the neutrino masses m_1 , m_2 , and m_3 are degenerate. If the lightest neutrino mass m_0 is small, one can see from Fig.1 that the value of $\sigma(|m_{ee}|)_{\max}/\sigma(m_0)$ in the inverted mass ordering case is much less than that in the normal mass ordering case.

The minimum value $\sigma(|m_{ee}|)_{\min}$ in the $\beta - \alpha$ plane can take the value zero if either $\text{Re}[PQ^*]/|PQ^*| = \cos(\text{Arg}(PQ^*)) = 0$ or $|Q| = 0$ is realized. There is a possibility that $|Q|$ becomes zero in the inverted mass ordering case, while that is impossible in the normal mass ordering case. The possibility $|Q| = 0$ will be also considered below.

3.2 Behavior of $\sigma(|m_{ee}|)$ according to the size of m_3 .

In this subsection, the behavior of $\sigma(|m_{ee}|)$ in the $\beta - \alpha$ plane is investigated according to the size of the lightest neutrino mass m_3 in the inverted mass ordering case. The expression of the maximum value $\sigma(|m_{ee}|)_{\max}$ is obtained in the preceding subsection. As discussed in subsection 2.2, we are especially interested in the case where $\sigma(|m_{ee}|)$ takes the value zero or approximately zero in the $\beta - \alpha$ plane. In the inverted mass ordering case, $\sigma(|m_{ee}|)$ can be zero not only if $\cos(\text{Arg}(PQ^*)) = 0$ but also if $|Q| = 0$. The behavior of $\sigma(|m_{ee}|)$ will be studied according to the size of m_3 in the following (A)(F).

3.2.1 (A): $m_3 \rightarrow 0$

In the limit $m_3 \rightarrow 0$, the factor $\partial|m_{ee}|/\partial m_3$ becomes

$$\frac{\partial|m_{ee}|}{\partial m_3} = m_3|Q| \frac{\text{Re}[PQ^*]}{|PQ^*|} \rightarrow |U_{e3}|^2 \frac{\text{Re}(m_3 Q^* P)}{\sqrt{\{\text{Re}(m_3 Q^* P)\}^2 + \{\text{Im}(m_3 Q^* P)\}^2}}. \quad (50)$$

On the plane curve in the $\beta - \alpha$ plane,

$$\frac{1}{|U_{e3}|^2} \text{Re}(m_3 Q^* P) = \sqrt{\Delta m_A^2} |U_{e1}|^2 \cos(-2\beta) + \sqrt{\Delta m_A^2 + \Delta m_\odot^2} |U_{e2}|^2 \cos(2\alpha - 2\beta) = 0, \quad (51)$$

or

$$\cos(-2\beta) + 0.44 \cos(2\alpha - 2\beta) = 0, \quad (52)$$

the factor $\partial|m_{ee}|/\partial m_3$ in Eq.(50) takes the minimum value zero,

$$\left(\frac{\partial|m_{ee}|}{\partial m_3} \right)_{\min} = \left(\frac{\partial|m_{ee}|}{\partial m_3} \right)_{\text{Re}(m_3 Q^* P)=0} = 0. \quad (53)$$

On the other hand, on the plane curve in the $\beta - \alpha$ plane,

$$\text{Im}(m_3 Q^* P) = \sqrt{\Delta m_A^2} |U_{e1}|^2 \sin(-2\beta) + \sqrt{\Delta m_A^2 + \Delta m_\odot^2} |U_{e2}|^2 \sin(2\alpha - 2\beta) = 0, \quad (54)$$

or

$$\sin(-2\beta) + 0.44 \sin(2\alpha - 2\beta) = 0, \quad (55)$$

the factor $\partial|m_{ee}|/\partial m_3$ takes the maximum value,

$$\left(\frac{\partial|m_{ee}|}{\partial m_3}\right)_{\max} = \left(\frac{\partial|m_{ee}|}{\partial m_3}\right)_{\text{Im}(m_3 Q^* P)=0} = |U_{e3}|^2 = 0.0227. \quad (56)$$

3.2.2 (B): $|U_{e3}|^2/m_3 \gg |U_{e1}|^2/m_1$ ($m_3 \ll \sqrt{\Delta m_A^2} |U_{e3}|^2/|U_{e1}|^2 \sim 0.002 \text{ eV}$)

For $m_3 \ll 0.002 \text{ eV}$, Q can be approximated by

$$Q \approx \frac{1}{m_3} |U_{e3}|^2 e^{2i\beta}, \quad (57)$$

which leads to

$$\frac{\partial|m_{ee}|}{\partial m_3} \approx |U_{e3}|^2 \frac{\text{Re}(m_3 Q^* P)}{\sqrt{\{\text{Re}(m_3 Q^* P)\}^2 + \{\text{Im}(m_3 Q^* P)\}^2}}. \quad (58)$$

On the plane curve,

$$\begin{aligned} & \frac{1}{|U_{e3}|^2} \text{Re}(m_3 Q^* P) \\ & \approx m_1 |U_{e1}|^2 \cos(-2\beta) + m_2 |U_{e2}|^2 \cos(-2\beta + 2\alpha) + m_3 |U_{e3}|^2 = 0, \end{aligned} \quad (59)$$

or

$$0.682 \sqrt{\Delta m_A^2} \cos(-2\beta) + 0.295 \sqrt{\Delta m_A^2 + \Delta m_\odot^2} \cos(-2\beta + 2\alpha) + 0.0227 m_3 = 0, \quad (60)$$

the factor becomes approximately zero $\partial|m_{ee}|/\partial m_3 \approx 0$. The inequality $m_3^2 \ll \Delta m_A^2$ has been used in driving Eq.(60). We therefore have

$$\sigma(|m_{ee}|)_{\min} = \sigma(|m_{ee}|)_{\text{Re}(m_3 Q^* P)=0} \approx 0. \quad (61)$$

On the other side, on the plane curve,

$$\frac{1}{|U_{e3}|^2} \text{Im}(m_3 Q^* P) = m_1 |U_{e1}|^2 \sin(-2\beta) + m_2 |U_{e2}|^2 \sin(-2\beta + 2\alpha) = 0, \quad (62)$$

or

$$0.682 \sqrt{\Delta m_A^2} \sin(-2\beta) + 0.295 \sqrt{\Delta m_A^2 + \Delta m_\odot^2} \sin(-2\beta + 2\alpha) = 0, \quad (63)$$

the factor $\partial|m_{ee}|/\partial m_3$ takes the maximum value $|U_{e3}|^2$. We have again used the inequality $m_3^2 \ll \Delta m_A^2$ in deriving Eq.(63). The maximum value of $\sigma(|m_{ee}|)$ is then given by

$$\sigma(|m_{ee}|)_{\max} = \sigma(|m_{ee}|)_{\text{Im}(m_3 Q^* P)=0} \approx |U_{e3}|^2 \sigma(m_3) = 0.0227 \sigma(m_3). \quad (64)$$

3.2.3 (C): $\frac{1}{m_1}|U_{e1}|^2 + \frac{1}{m_2}|U_{e2}|^2 - \frac{1}{m_3}|U_{e3}|^2 = 0$ ($m_3 = 0.0012 \text{ eV}$)

In this case, $Q = 0$ at the point $(\beta, \alpha) = (\pi/2, 0)$ in the $\beta - \alpha$ plane. Because $\partial|m_{ee}|/\partial m_3 = 0$ at this point from Eq.(47), one has

$$\sigma(|m_{ee}|)_{\min} = \sigma(|m_{ee}|)_{(\beta, \alpha)=(\pi/2, 0)} = 0. \quad (65)$$

The maximum value of $\sigma(|m_{ee}|)$ in the $\beta - \alpha$ plane becomes

$$\sigma(|m_{ee}|)_{\max} = \sigma(|m_{ee}|)_{(\beta, \alpha)=(0, 0)} = \left\{ 2|U_{e3}|^2 \right\} \sigma(m_3) = 0.0454 \sigma(m_3). \quad (66)$$

3.2.4 (D): $\frac{1}{m_1}|U_{e1}|^2 - \frac{1}{m_2}|U_{e2}|^2 - \frac{1}{m_3}|U_{e3}|^2 = 0$ ($m_3 = 0.0029 \text{ eV}$)

In this case, $Q = 0$ at the point $(\beta, \alpha) = (\pi/2, \pi/2)$. Since $\partial|m_{ee}|/\partial m_3 = 0$ at this point from Eq.(47), we have

$$\sigma(|m_{ee}|)_{\min} = \sigma(|m_{ee}|)_{(\beta, \alpha)=(\pi/2, \pi/2)} = 0. \quad (67)$$

The maximum value of $\sigma(|m_{ee}|)$ becomes

$$\sigma(|m_{ee}|)_{\max} = \sigma(|m_{ee}|)_{(\beta, \alpha)=(0, 0)} = \left\{ 2 \frac{m_3}{m_1} |U_{e1}|^2 \right\} \sigma(m_3) = 0.079 \sigma(m_3). \quad (68)$$

3.2.5 (E): $\frac{1}{m_2}|U_{e2}|^2 \gg \frac{1}{m_3}|U_{e3}|^2$ **and** $m_3^2 \lesssim \Delta m_A^2$ ($0.004 \text{ eV} \ll m_3 \lesssim 0.05 \text{ eV}$)

In this case, we obtain the relation, $|U_{e2}|^2 \gg (m_2/m_3)|U_{e3}|^2 > (m_3/m_2)|U_{e3}|^2$, which allows us to use the approximation,

$$\begin{aligned} P &\approx m_1|U_{e1}|^2 + m_2|U_{e2}|^2 e^{2i\alpha}, \\ Q &\approx \frac{1}{m_1}|U_{e1}|^2 + \frac{1}{m_2}|U_{e2}|^2 e^{2i\alpha}. \end{aligned} \quad (69)$$

The factor $\partial|m_{ee}|/\partial m_3$ is then approximated,

$$\frac{\partial|m_{ee}|}{\partial m_3} \approx \frac{m_3 \left\{ |U_{e1}|^4 + |U_{e2}|^4 + \left(\frac{m_2}{m_1} + \frac{m_1}{m_2} \right) |U_{e1}|^2 |U_{e2}|^2 \cos 2\alpha \right\}}{(m_1|U_{e1}|^2)^2 + (m_2|U_{e2}|^2)^2 + 2(m_1|U_{e1}|^2)(m_2|U_{e2}|^2) \cos 2\alpha}. \quad (70)$$

The factor $\partial|m_{ee}|/\partial m_3$ scarcely depends on the Majorana phase β , and is a monotone decreasing function of the Majorana phase α ($0 \leq \alpha \leq \pi/2$). The $\partial|m_{ee}|/\partial m_3$ takes the maximum value $(m_3/m_1)|U_{e1}|^2 + (m_3/m_2)|U_{e2}|^2 + O(|U_{e3}|^2)$ at $\alpha = 0$, and the minimum value $(m_3/m_1)|U_{e1}|^2 - (m_3/m_2)|U_{e2}|^2 + O(|U_{e3}|^2)$ at $\alpha = \pi/2$. In other words, $\sigma(|m_{ee}|)$ scarcely depends on β , and is a monotone decreasing function of α ($0 \leq \alpha \leq \pi/2$). The $\sigma(|m_{ee}|)$ takes the maximum value at $\alpha = 0$,

$$\begin{aligned} \sigma(|m_{ee}|)_{\max} &= \sigma(|m_{ee}|)_{\alpha=0} \\ &\approx \left\{ \frac{m_3}{m_1}|U_{e1}|^2 + \frac{m_3}{m_2}|U_{e2}|^2 \right\} \sigma(m_3) \\ &= \left\{ 0.977 \frac{m_3}{\sqrt{m_3^2 + \Delta m_A^2}} \right\} \sigma(m_3), \end{aligned} \quad (71)$$

and the minimum value at $\alpha = \pi/2$,

$$\begin{aligned}\sigma(|m_{ee}|)_{\min} &= \sigma(|m_{ee}|)_{\alpha=\pi/2} \\ &\approx \left\{ \frac{m_3}{m_1} |U_{e1}|^2 - \frac{m_3}{m_1} |U_{e2}|^2 \right\} \sigma(m_3) \\ &= \left\{ 0.387 \frac{m_3}{\sqrt{m_3^2 + \Delta m_A^2}} \right\} \sigma(m_3),\end{aligned}\tag{72}$$

where we have used the approximation $m_1 \approx m_2$, and neglected the term of the order $O(|U_{e3}|^2)$ in $\partial|m_{ee}|/\partial m_3$.

3.2.6 (F): $m_3^2 \gg \Delta m_A^2$ such that $m_1 \approx m_2 \approx m_3$ ($m_3^2 \gg (0.05\text{eV})^2$)

Using the relation $Q \approx P/m_3^2$, we have

$$\frac{\partial |m_{ee}|}{\partial m_3} \approx \frac{|m_{ee}|}{m_3},\tag{73}$$

or

$$\frac{\sigma(|m_{ee}|)}{|m_{ee}|} \approx \frac{\sigma(m_3)}{m_3}.\tag{74}$$

As in the case of $m_1 \approx m_2 \approx m_3$ in the normal mass ordering, the relative error of the effective mass $|m_{ee}|$ has the same magnitude as that of the lightest neutrino mass m_3 , and the following relations hold,

$$\begin{aligned}\sigma(|m_{ee}|)_{\max} &= \sigma(|m_{ee}|)_{(\beta,\alpha)=(0,0)} \approx \sigma(m_3), \\ \sigma(|m_{ee}|)_{\min} &= \sigma(|m_{ee}|)_{(\beta,\alpha)=(\pi/2,\pi/2)} \approx 0.364 \sigma(m_3).\end{aligned}\tag{75}$$

4 Minimum condition of $\sigma(m_0)$ for probing the Majorana phases.

What values the Majorana phases β, α take is an interesting subject. When the values of β and α are changed, the effective mass $|m_{ee}|$ varies from $|m_{ee}|_{\min}$ to $|m_{ee}|_{\max}$, where we have denoted the maximum value of $|m_{ee}|$ in the $\beta - \alpha$ plane as $|m_{ee}|_{\max}$ and the minimum value as $|m_{ee}|_{\min}$. Up to now, only the upper limit of the effective mass $|m_{ee}|$ is reported in experiments, as well as the upper limit of the lightest neutrino mass is. Assuming that the lightest neutrino mass m_0 is measured $m_0 \pm \sigma(m_0)$, we studied in section 2 and 3 how the predicted value of $|m_{ee}|$ is given. The predicted value of $|m_{ee}|$,

$$|m_{ee}| \pm n \sigma(|m_{ee}|), \quad (n = 1, 2, 3, \dots),\tag{76}$$

depends on the values of Majorana phases β and α . The larger the uncertainty $\sigma(m_0)$ is, the larger becomes the uncertainty $\sigma(|m_{ee}|)$ of the predicted value of $|m_{ee}|$. If the

uncertainty $\sigma(|m_{ee}|)$ is large such that it exceeds the order of $(|m_{ee}|_{\max} - |m_{ee}|_{\min})$, one can not obtain the information on the Majorana phases β and α through $|m_{ee}|$. In this section, we study the minimum condition which the relative error of the lightest neutrino mass m_0 needs to satisfy, in order to obtain some information on the Majorana phases β and α through the effective mass $|m_{ee}|$.

Let us pay attention to the maximum value $\sigma(|m_{ee}|)_{\max}$ of $\sigma(|m_{ee}|)$ in the $\beta - \alpha$ plane and the point (β, α) at which $\sigma(|m_{ee}|)_{\max}$ is realized. If the predicted value of $|m_{ee}|$ at this point, $|m_{ee}| \pm n \sigma(|m_{ee}|)_{\max}$, satisfies the following relations simultaneously,

$$\begin{aligned} |m_{ee}|_{\max} &\leq |m_{ee}| + n \sigma(|m_{ee}|)_{\max}, \\ |m_{ee}| - n \sigma(|m_{ee}|)_{\max} &\leq |m_{ee}|_{\min}, \end{aligned} \quad (77)$$

the information on the Majorana phases β and α is not obtained effectively, because the allowed region of the predicted value of $|m_{ee}|$ is too wide. As discussed in section 2 and 3, $\sigma(|m_{ee}|)$ takes its maximum value $\sigma(|m_{ee}|)_{\max}$ at the point $(\beta, \alpha) = (0, 0)$, at which the effective mass $|m_{ee}|$ takes its maximum value $|m_{ee}|_{\max}$. One therefore obtains from Eq.(77),

$$|m_{ee}|_{\max} - n \sigma(|m_{ee}|)_{\max} \leq |m_{ee}|_{\min}, \quad (78)$$

or

$$\frac{1}{n} \leq \frac{\sigma(|m_{ee}|)_{\max}}{|m_{ee}|_{\max} - |m_{ee}|_{\min}}. \quad (79)$$

When $\sigma(|m_{ee}|)_{\max}$ has large value such as Eq.(79), we can not obtain the information on the Majorana phases β and α effectively because of the wide allowed region of the predicted value of $|m_{ee}|$. The maximum value $\sigma(|m_{ee}|)_{\max}$ is represented by the uncertainty $\sigma(m_0)$ of the lightest neutrino mass m_0 if one put the uncertainties of the oscillation parameters zero. We can therefore restrict the uncertainty $\sigma(m_0)$ of m_0 in order to obtain the information on the Majorana phases β and α . In the following subsections, the uncertainty $\sigma(m_0)$ is restricted in the normal mass ordering case and the inverted mass ordering case, respectively.

4.1 In the normal mass ordering case.

In this subsection dealing with the normal mass ordering case, we consider the lightest neutrino mass m_1 that satisfies

$$m_2|U_{e2}|^2 + m_3|U_{e3}|^2 < m_1|U_{e1}|^2, \quad (80)$$

or

$$m_1 > 0.0063 \text{ eV}. \quad (81)$$

Since the lower limit of m_1 is not given by experiments, rather large value of m_1 is interesting for the present. The effective mass $|m_{ee}|$ takes its maximum value at the point $(\beta, \alpha) = (0, 0)$,

$$|m_{ee}|_{\max} = m_1|U_{e1}|^2 + m_2|U_{e2}|^2 + m_3|U_{e3}|^2, \quad (82)$$

while it takes its minimum value at the point $(\beta, \alpha) = (\pi/2, \pi/2)$ [18, 22],

$$|m_{ee}|_{\min} = m_1|U_{e1}|^2 - m_2|U_{e2}|^2 - m_3|U_{e3}|^2, \quad (83)$$

for m_1 satisfying Eq.(80). The width between $|m_{ee}|_{\max}$ and $|m_{ee}|_{\min}$ then becomes

$$|m_{ee}|_{\max} - |m_{ee}|_{\min} = 2(m_2|U_{e2}|^2 + m_3|U_{e3}|^2). \quad (84)$$

Using Eq.(84) and Eq.(17), we get

$$\frac{\sigma(|m_{ee}|)_{\max}}{|m_{ee}|_{\max} - |m_{ee}|_{\min}} = \frac{m_1 \left\{ |U_{e1}|^2 + \frac{m_1}{m_2}|U_{e2}|^2 + \frac{m_1}{m_3}|U_{e3}|^2 \right\}}{2(m_2|U_{e2}|^2 + m_3|U_{e3}|^2)} \times \left\{ \frac{\sigma(m_1)}{m_1} \right\}. \quad (85)$$

In the right-handed side of this equation, the coefficient of $\sigma(m_1)/m_1$ is a monotone increasing function of the lightest neutrino mass m_1 ($> 0.0065\text{eV}$). Let us consider two concrete cases; one is the case where $m_1 \gg \sqrt{\Delta m_A^2} \sim 0.05\text{eV}$, and the other is the case where $m_1 = \sqrt{\Delta m_A^2} \sim 0.05\text{eV}$. In the first case, $m_1 \gg \sqrt{\Delta m_A^2} \sim 0.05\text{eV}$, three neutrino masses are degenerate, $m_1 \approx m_2 \approx m_3$. When $n = 2$, Eq.(79) becomes

$$\frac{1}{2} \leq 1.57 \left\{ \frac{\sigma(m_1)}{m_1} \right\}, \quad (86)$$

which implies that if the relative error $\sigma(m_1)/m_1$ is larger than 32%, the relation $|m_{ee}|_{\max} - 2\sigma(|m_{ee}|)_{\max} \leq |m_{ee}|_{\min}$ (Eq.(78)) holds, and the information on the Majorana phases β and α is not obtained effectively. Thereby, in the case $m_1 \gg \sqrt{\Delta m_A^2} \sim 0.05\text{eV}$, the minimum condition so as to raise the possibility of obtaining the information on β and α is that the relative error $\sigma(m_1)/m_1$ of the lightest neutrino mass should be less than 32% at least. In the second case, $m_1 = \sqrt{\Delta m_A^2} \sim 0.05\text{eV}$, Eq.(79) with $n = 2$ becomes

$$\frac{1}{2} \leq 1.48 \left\{ \frac{\sigma(m_1)}{m_1} \right\}. \quad (87)$$

Therefore, in the case $m_1 = \sqrt{\Delta m_A^2} \sim 0.05\text{eV}$, the minimum condition so as to raise the possibility of obtaining the information on β and α is that the relative error $\sigma(m_1)/m_1$ should be less than 34% at least.

4.2 In the inverted mass ordering case.

The effective mass $|m_{ee}|$ takes the maximum value at the point $(\beta, \alpha) = (0, 0)$,

$$|m_{ee}|_{\max} = m_1|U_{e1}|^2 + m_2|U_{e2}|^2 + m_3|U_{e3}|^2, \quad (88)$$

while it takes the minimum value at the point $(\beta, \alpha) = (\pi/2, \pi/2)$,

$$|m_{ee}|_{\min} = m_1|U_{e1}|^2 - m_2|U_{e2}|^2 - m_3|U_{e3}|^2, \quad (89)$$

and the width becomes

$$|m_{ee}|_{\max} - |m_{ee}|_{\min} = 2(m_2|U_{e2}|^2 + m_3|U_{e3}|^2). \quad (90)$$

From this equation and Eq.(49), one has

$$\frac{\sigma(|m_{ee}|)_{\max}}{|m_{ee}|_{\max} - |m_{ee}|_{\min}} = \frac{m_3 \left\{ \frac{m_3}{m_1}|U_{e1}|^2 + \frac{m_3}{m_2}|U_{e2}|^2 + |U_{e3}|^2 \right\}}{2(m_2|U_{e2}|^2 + m_3|U_{e3}|^2)} \times \left\{ \frac{\sigma(m_3)}{m_3} \right\}. \quad (91)$$

In the right-handed side of this equation, the coefficient of $\sigma(m_3)/m_3$ is a monotone increasing function of the lightest neutrino mass m_3 . Two concrete cases are considered here; one is $m_3 \gg \sqrt{\Delta m_A^2} \sim 0.05\text{eV}$, and the other is $m_3 = \sqrt{\Delta m_A^2} \sim 0.05\text{eV}$. In the first case, three neutrino masses are degenerate and Eq.(79) with $n = 2$ becomes

$$\frac{1}{2} \leq 1.57 \left\{ \frac{\sigma(m_3)}{m_3} \right\}. \quad (92)$$

Hence, in the case $m_3 \gg 0.05\text{eV}$, the minimum condition so as to raise the possibility of obtaining the information on β and α is that the relative error $\sigma(m_3)/m_3$ of the lightest neutrino mass m_3 should be less than 32% at least. The above result with the lightest neutrino mass $m_3 \gg \sqrt{\Delta m_A^2}$ in the inverted mass ordering case is the same as that with the lightest neutrino mass $m_1 \gg \sqrt{\Delta m_A^2}$ in the normal mass ordering case. This is natural because three neutrino masses are degenerate $m_1 \approx m_2 \approx m_3$ in both cases. In the second case, $m_3 = \sqrt{\Delta m_A^2} \sim 0.05\text{eV}$, Eq.(79) with $n = 2$ becomes

$$\frac{1}{2} \leq 0.81 \left\{ \frac{\sigma(m_3)}{m_3} \right\}. \quad (93)$$

Then, in the case $m_3 \sim 0.05\text{eV}$, the minimum condition so as to raise the possibility of obtaining the information on β and α is that the relative error $\sigma(m_3)/m_3$ should be less than 62% at least.

We shall make two comments here. The first comment is concerned with the conditions Eq.(93) and Eq.(87) when the lightest neutrino mass is about 0.05eV. Eq.(93) shows that the relative error of the lightest neutrino mass should be less than 62% in the inverted mass ordering case, while Eq.(87) shows that the relative error of the lightest neutrino mass should be less than 34% in the normal mass ordering case. The contrast between these comes mainly from the fact that the behavior of $\sigma(|m_{ee}|)_{\max}/\sigma(m_0)$ in the inverted mass ordering case is different from that in the normal mass ordering case. As seen from Fig.1, the value of $\sigma(|m_{ee}|)_{\max}/\sigma(m_0)$ in the inverted mass ordering case is considerably less than the value of that in the normal mass ordering case in the region $m_0 \lesssim 0.05\text{eV}$. The second comment is about the minimum condition for the relative error $\sigma(m_0)/m_0$ to raise the possibility of obtaining the information on β and α . In this paper, we have set the errors of the oscillation parameters zero. When these errors are included in our calculation, the condition required for the relative error of the lightest neutrino mass m_0 becomes stricter. Hence, the condition for $\sigma(m_0)/m_0$ obtained in this chapter should be regarded as the least necessary condition.

5 Conclusion

We studied the influence of the error of the lightest neutrino mass m_0 on the predicted value of the effective Majorana mass $|m_{ee}|$ under the assumption that the lightest neutrino mass is measured in experiments. The case of assuming the normal mass ordering and the case of assuming the inverted mass ordering are studied, respectively. The error of the predicted value of the effective Majorana mass $\sigma(|m_{ee}|)$ is represented by the error of the lightest neutrino mass $\sigma(m_0)$ with the law of propagation of errors provided that one sets the errors of the oscillation parameters zero, respectively. We have studied the error $\sigma(|m_{ee}|)$ depending on the Majorana phases β and α how it behaves in the $\beta - \alpha$ plane, and how it changes according to the size of the lightest neutrino mass m_0 . It is shown that the maximum value of $\sigma(|m_{ee}|)$ in the $\beta - \alpha$ plane does not exceed the value of $\sigma(m_0)$ in general, and that the minimum value of $\sigma(|m_{ee}|)$ in the $\beta - \alpha$ plane can be zero when the lightest neutrino mass m_0 is small. An interesting feature is seen when the lightest neutrino mass m_0 has large value $m_0^2 \gg \Delta m_A^2$ so that three neutrino masses are degenerate. In this case, the relative error of $|m_{ee}|$ and that of m_0 are the same. We also investigated how the maximum value of $\sigma(|m_{ee}|)$ in the $\beta - \alpha$ plane changes according to the size of the lightest neutrino mass m_0 . In the normal mass ordering case, the maximum value of $\sigma(|m_{ee}|)$ in the $\beta - \alpha$ plane is almost the same as $\sigma(m_0)$ when the lightest neutrino mass $m_0 = m_1$ is large such as $m_1^2 \gg \Delta m_A^2$. When the value of $m_0 = m_1$ decreases, the maximum value of $\sigma(|m_{ee}|)$ decreases gradually, and it becomes $|U_{e1}|^2 \sigma(m_0) \sim 0.682 \sigma(m_0)$ if m_1 is approximately zero. In the inverted mass ordering case, the maximum value of $\sigma(|m_{ee}|)$ in the $\beta - \alpha$ plane is almost the same as $\sigma(m_0)$ when the lightest neutrino mass $m_0 = m_3$ is large such as $m_3^2 \gg \Delta m_A^2$. When the value of $m_0 = m_3$ decreases, however, the maximum value of $\sigma(|m_{ee}|)$ decreases rapidly, and it becomes $|U_{e3}|^2 \sigma(m_0) \sim 0.0227 \sigma(m_0)$ if m_3 is approximately zero.

In Eq.(8) where $\sigma(|m_{ee}|)$ is represented by $\sigma(m_0)$ (the case of $m_0 = m_1$), the factor $\partial|m_{ee}|/\partial m_0$ ($m_0 = m_1$) has been calculated straightforwardly and the result Eq.(9) contains $|m_{ee}|$ in the denominator. When $|m_{ee}|$ becomes very small, therefore, one may guess that $\sigma(|m_{ee}|)$ can take very large value at first sight. However, it is shown by the analytic calculation that the maximum value of $\sigma(|m_{ee}|)$ in the $\beta - \alpha$ plane does not exceed $\sigma(m_0)$. This demonstration owes the complex quantities P and Q which are introduced in Eq.(13) and Eq.(14). Utilizing these P and Q , we showed that the maximum value of $\sigma(|m_{ee}|)$ in the $\beta - \alpha$ plane is given by Eq.(17) in the normal mass ordering case, and by Eq.(49) in the inverted mass ordering case, respectively.

We have especially paid attention to the influence of the error of the lightest neutrino mass $\sigma(m_0)$ on the uncertainty of the predicted value of the effective Majorana mass $\sigma(|m_{ee}|)$. If we wish to have the information on the Majorana phases β and α through $|m_{ee}|$, we need to investigate the dependence of $|m_{ee}|$ on β and α . Hence, we studied the uncertainty $\sigma(|m_{ee}|)$ of the predicted value of $|m_{ee}|$ in detail; how does it behave in the $\beta - \alpha$ plane, how does it depend on the lightest neutrino mass m_0 , and

how large is it. At the outset, the minimum condition required for the relative error $\sigma(m_0)/m_0$ in order to raise the possibility of obtaining the information on β and α is considered in section 4. For instance, in the case of $m_0^2 \gg \Delta m_A^2$ ($m_1 \approx m_2 \approx m_3$), the relative error of m_0 should be less than 32% so that the allowed region of $|m_{ee}|$ does not cover the width ($|m_{ee}|_{\max} - |m_{ee}|_{\min}$) by 2σ .

Hereafter, it will be necessary to study minimum condition required for the relative error of m_0 so as to narrow the allowed region of the predicted value $|m_{ee}|$. In the following, we give a simple example in the case $m_0 = m_1 \approx m_2 \approx m_3$. If the CP-invariance are maintained, the Majorana phases take the values, $\alpha = n\pi/2$, $\beta = n'\pi/2$, ($n, n' \in \mathbf{Z}$), and in the restricted region $-\pi/2 < \beta, \alpha \leq \pi/2$,

$$(\beta, \alpha) = (0, 0), \quad (\frac{\pi}{2}, 0), \quad (0, \frac{\pi}{2}), \quad (\frac{\pi}{2}, \frac{\pi}{2}). \quad (94)$$

The values of $|m_{ee}|$ at the above four points (β, α) satisfy

$$|m_{ee}|_{(\beta, \alpha)=(0,0)} > |m_{ee}|_{(\beta, \alpha)=(\frac{\pi}{2}, 0)} > |m_{ee}|_{(\beta, \alpha)=(0, \frac{\pi}{2})} > |m_{ee}|_{(\beta, \alpha)=(\frac{\pi}{2}, \frac{\pi}{2})}. \quad (95)$$

The allowed region of the predicted value of $|m_{ee}|_{(\beta, \alpha)}$ is given by

$$|m_{ee}|_{(\beta, \alpha)} \pm n \sigma(|m_{ee}|)_{(\beta, \alpha)}, \quad (n = 1, 2, 3, \dots). \quad (96)$$

If the error $\sigma(m_0)$ of the lightest neutrino mass is large such as

$$|m_{ee}|_{(\frac{\pi}{2}, 0)} - n \sigma(|m_{ee}|)_{(\frac{\pi}{2}, 0)} \leq |m_{ee}|_{(0, \frac{\pi}{2})} + n \sigma(|m_{ee}|)_{(0, \frac{\pi}{2})}, \quad (97)$$

the allowed region of $|m_{ee}|_{(\frac{\pi}{2}, 0)}$ and that of $|m_{ee}|_{(0, \frac{\pi}{2})}$ overlap. Under this condition, the information on the CP violating Majorana phases β and α which lie between $|m_{ee}|_{(\frac{\pi}{2}, 0)}$ and $|m_{ee}|_{(0, \frac{\pi}{2})}$ can not be obtained. Using the relation $\sigma(|m_{ee}|) = |m_{ee}| \{\sigma(m_0)/m_0\}$, Eq.(97) becomes

$$\frac{\sigma(m_0)}{m_0} \geq \frac{1}{n} \left(\frac{|U_{e2}|^2 - |U_{e3}|^2}{|U_{e1}|^2} \right) = 0.20, \quad (98)$$

where $n = 2$ has been adopted. We therefore see that, in order to obtain the information on β and α through the $|m_{ee}|$ ($|m_{ee}|_{(\frac{\pi}{2}, 0)} < |m_{ee}| < |m_{ee}|_{(0, \frac{\pi}{2})}$) with CP violating Majorana phases β, α , the relative error of m_0 should be less than 20% at least. This is a simple example, and further investigation will be needed. Lastly, we note that we have set the errors of the measured values of the oscillation parameters zero for simplicity. Although these errors are expected to decrease by future experiments, it is necessary to consider the influence of these errors of the oscillation parameters on the $|m_{ee}|$.

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